Equivariant Cohomology and Quiver Varieties

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About This Course

These are typed notes from a seminar led by Andrey Smirnov on Equivariant Cohomology and Quiver Varieties. Each week was a lecture given by one of the graduate students of the seminar on a topic regarding this area. I've typed up my written notes from the seminar, and any mistakes here are mine and not the speakers.

Contents

1 1/24: Intro to Cohomology

Consider a chain complex

$$\cdots \to C_{n+1} \to C_n \to C_{n-1} \to \cdots$$

Where $\partial : C_{i+1} \to C_i$. Define $C_n^* = \mathrm{Hom}(C_n, G)$ for some group G of coefficients. Then $\delta : C_{n-1}^* \to C_n^*$

Definition.

$$H^n(C, G) = \ker(\delta)/\mathrm{im}(\delta)$$

Definition. The cohomology ring is defined to be

$$H^*(X, R) = \bigoplus_{n\in\mathbb{Z}} H^n(X, R)$$

Lets see some examples of computations:

Example. If X={pt}, then $\sigma : \Delta^0 \to X$ maps $\{0\} \mapsto \{x\}$, then

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{else} \end{cases}$$

So

$$H^i(X) = \mathrm{Hom}(H_i(X), \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{else} \end{cases}$$

A general idea to keep in mind is that for any topological space X, to compute $H_i(X)$ we always triangulate into simplices: A face is a 2-dim simplex, a line is a 1-dim simplex, and a point is a 0-dim simplex. From these simplices we form vector spaces that make up our chain complex:

C_2 is the vector space (say over $\mathbb{C}$) with basis of 2-dim simplices, C_1 is the complex vector space with basis given by 1-dim simplices, and C_0 is the vector space given by 0-dim simplices. There could be more, but if we're working with a 2-dim topological manifold, this is all we have.

Example. To compute the boundary operator of, say, a 2-simplex we need to orient and label the vertices and edges. Say we have the following 2-simplex: INSERT PICTURE HERE!!!!!!!!!!!!

If we call this simplex Δ, then computing the boundary operator is straight forward: We simply add up all the 1-simplices corresponding to their correct orientation:

$$\partial(\Delta) = [b, c] + [c, a] + [a, b]$$

Where each of these $[b, c], [c, a], [a, b] \in C_1$, and as oriented simplices we have to remind ourselves that $[b, c] = -[c, b]$

We can sup up this example to higher dimensional simplices: $[a, b, c]$ represents an oriented 2-simplex, maybe a triangle with vertices {a,b,c}, $[a, b, c, d]$ represents an oriented 3-dimensional simplex with oriented vertices, INSERT PICTURES!!!!!!!!!!!!!!!!!!

In general we have the formula:

$$\partial[i_1, i_2, ..., i_n] = \sum_{k=1}^{n} (-1)^k [i_1, i_2, .., \not{i_k}, .., i_n]$$

Where $\not{i_k}$ means remove the i_k^{th} entry.

Example. If we know that

$$\partial[a, b, c] = -[b, c] + [a, c] - [a, b]$$

What is is corresponding 2-simplex that this represents? INSERT PICTURE!!!!!!!!!!!!!!!!!!!

Example. Lets look at some well-defined topological spaces. First S^1. Here is a certain triangulation of the circle: INSERT PICTURE!!!!!!!!!!!!!!!

What are the C_i's in this case? S^1 is 1-dimensional so C_i for $i \geq 2$ and $i \leq -1$ are all 0. In the above triangulation we have 3 "vertices" and 3 "edges" so C_1 and C_0 are both 3 dimensional. This means $C_1 = \mathbb{C}^3$ and $C_0 = C^3$ where $C_1 = \mathbb{C}^3([a, b], [b, c], [c, a])$ and $C_0 = \mathbb{C}^3(a, b, c)$.

The main point of all this is that homology and cohomology are topological INVARIANTS. Meaning the choice of triangulation doesn't matter. Let's now compute the chain complex, and boundary maps explicitly[1]

$$\partial[a, b] = -b + a$$
$$\partial[b, c] = -c + b$$
$$\partial[c, a] = -a + c$$

As a result we have the following chain complex:

$$0 \to \mathbb{C}^3 \to \mathbb{C}^3 \to 0$$

Where the first map is ∂_0, the second is ∂_1, and ∂_1 has a matrix representation (if we choose basis [a,b], [b,c],[c,a]) as follows:

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

From this we compute the homology of this space:

$$H_1 = \ker \partial_1 / \operatorname{im} \partial_0 = \mathbb{C}/\{0\} \simeq \mathbb{C}$$

$$H_2 = \ker \partial_2 / \operatorname{im} \partial_1 = \mathbb{C}^3/\mathbb{C}^2 \simeq \mathbb{C}$$

Notice further that in the circle we could choose the "trivial" triangulation consisting of a single point and 1 line. If we call this point "a" then the chain complex is

$$C_0 = \mathbb{C} \to \mathbb{C} = C_1$$

and the differential map is trivial since $\partial = a - a = 0$

This just goes to show why we call homology and cohomology INVARIANTS, the choice of triangulation doesn't matter.

[1] Andrey asks "You guys know linear algebra?"

Example. Consider the space $S^2 \simeq \mathbb{CP}^1$

One example of a triangulation is to remove the north pole (NP) and consider the chain complex $C_2 = \mathbb{C}, C_0 = \{NP\}$. Then we get

$$0 \to \mathbb{C} \to 0 \to \mathbb{C} \to 0$$

We can notice immediately that we can read off the homology groups: $H_0 = \mathbb{C}, H_2 = \mathbb{C}$

In general to compute projective space, over the reals or complex numbers is done in this fashion. Let's describe this space geometrically with some pretty pictures.

Example. $\mathbb{RP}^1$ = lines in $\mathbb{R}^2 \simeq S^2$

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Example. $\mathbb{RP}^2$ = lines in $\mathbb{R}^3$

INSERT PICTURE!!!!!!!!!!!!!!!!!11

Example. $\mathbb{RP}^3$

INSERT PICTURE!!!!!!!!!!!!!!!!!11

2 1/31: Examples of Cohomology of $\mathbb{P}^n$ and $Gr(k, n)$

Lets explicitly describe $\mathbb{CP}^n$. This manifold is equivalence classes of lines:

$$\mathbb{CP}^n = \{[z_0, z_1, ..., z_n] : (z_0, ..., z_n) \neq 0, z_i \in \mathbb{C}\}$$

We can choose a decomposition of this space as follows

3 2/21: Finishing up Grassmannian and Starting Equivariant Cohomology

4 2/28: Equivariant Cohomology of $\mathbb{P}^1, \mathbb{P}^2$

The action of $(\mathbb{C}^\times)^2$ on $\mathbb{P}^1$ is given by

$$(t_1, t_2) \cdot [z_0 : z_1] = [t_1 z_0 : t_2 z_1].$$

Let us consider the case when $[z_0 : z_1] \neq [1 : 0]$. Then, $z_0 \neq 0$ or $z_1 \neq 0$. If $z_0 \neq 0$, we can set $t_1 = 1$ and $t_2 = z_1/z_0$ to obtain

$$(t_1, t_2) \cdot [z_0 : z_1] = [t_1 z_0 : t_2 z_1] = [z_0 : z_1 \cdot z_1/z_0] = [z_0^2/z_1 : z_1].$$

Similarly, if $z_1 \neq 0$, we can set $t_1 = z_0/z_1$ and $t_2 = 1$ to obtain

$$(t_1, t_2) \cdot [z_0 : z_1] = [t_1 z_0 : t_2 z_1] = [z_0 \cdot z_0/z_1 : z_1] = [z_0 : z_1^2/z_0].$$

Therefore, the $(\mathbb{C}^\times)^2$-orbit of $[z_0 : z_1]$ in $\mathbb{P}^1$ is either $[z_0 : z_1^2/z_0]$ or $[z_0^2/z_1 : z_1]$ if $[z_0 : z_1] \neq [1 : 0]$. The orbit of $[1 : 0]$ is just the fixed point $[1 : 0]$ itself.

In summary, the orbits of the action of $(\mathbb{C}^\times)^2$ on $\mathbb{P}^1$ are

$$[1:0], \quad [z_0 : z_1^2/z_0], \quad [z_0^2/z_1 : z_1]$$

for $[z_0 : z_1] \neq [1:0]$.

for $(t_1, t_2) \in (\mathbb{C}^\times)^2$.

The one-dimensional orbits of the action of $(\mathbb{C}^\times)^2$ on $\mathbb{P}^1$ are the orbits of points that are not fixed by the action. The only fixed point is $[1:0]$, so we only need to consider the one-dimensional orbits of points of the form $[z_0 : z_1]$ with $z_0 \neq 0$ and $z_1 \neq 0$.

Let $[z_0 : z_1]$ be a point of $\mathbb{P}^1$ with $z_0 \neq 0$ and $z_1 \neq 0$. The orbit of $[z_0 : z_1]$ under the action of $(\mathbb{C}^\times)^2$ is given by

Therefore, the one-dimensional orbit of $[z_0 : z_1]$ is given by the set of all points of the form $[\lambda z_0 : z_1]$ for $\lambda \in \mathbb{C}^\times$. Since the projective line $\mathbb{P}^1$ is compact, the orbit of any point is a closed subset of $\mathbb{P}^1$, and the one-dimensional orbit of $[z_0 : z_1]$ is precisely the projective line passing through $[z_0 : z_1]$ and $[1:0]$. Therefore, there is one one-dimensional orbit for each point $[z_0 : z_1]$ with $z_0 \neq 0$ and $z_1 \neq 0$.

In summary, the one-dimensional orbits of the action of $(\mathbb{C}^\times)^2$ on $\mathbb{P}^1$ are precisely the projective lines passing through the point $[1:0]$ and each point $[z_0 : z_1]$ with $z_0 \neq 0$ and $z_1 \neq 0$ in $\mathbb{P}^1$.

5 3/7: Equivariant Cohomology of $\mathbb{P}^n$, Grassmannians, Flag Varieties

6 3/28: Equivariant Integration

Say we have a torus acting on a variety $T = (\mathbb{C}^\times)^n \curvearrowright X$, with the set of fixed points being finite: $|X^T| < \infty$. If we take an equivariant cohomology class $\omega \in H_T^*(X)$ the formula for integration over X is as follows:

$$\int_X \omega = \sum_{p \in X^T} \frac{\omega|_p}{e(T_pX)}$$

Where $\omega|_p \in H_T^*(pt) = \mathbb{C}[u_1, ..., u_n]$, and $e(T_pX)$ is the Euler class of the tangent space to p.

We need to explain a bit about the Euler class. T gives an action on T_pX which is a representation of the torus. At each fixed point, we have various weights of T_pX, and in a neighborhood of T_pX the character at p of T_pX is the sum of directions of repelling/attracting characters

INSERT PICTURE!!!!!!!!!!!!!!!!!!!!!!!!!!

I.e. if our torus action has coordinates $\alpha_1, ..., \alpha_{dimX}$, then $\text{char}_p(T_pX) = \alpha_1 + \cdots + \alpha_{dimX}$ is the character of the action at the fixed point (a sum of weights)

Example. For the action of $(\mathbb{C}^\times)^2 \curvearrowright \mathbb{C}^2$ gotten by scaling the coordinate vectors by u_1, u_2

INSERT PICTURE!!!!!!!!!!!!!!!!!!!!!!!!!!

Then the character at the unique fixed point 0 is u_1, u_2

From here we define the Euler class to be the product of characters:

$$e(T_pX) = \prod_{i=1}^{\dim X} \alpha_i$$

Example.

$$\int_{\mathbb{P}^1} c_1 = 1$$

Here we consider the variety $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$ which is topologically a 2-sphere, where $\mathbb{C}^2$ an action of the 2-dimensional torus with weights u_1, u_2:

$$\mathbb{C}^\times_{u_1} \times \mathbb{C}^\times_{u_2} \curvearrowright \mathbb{C}^2$$

Gotten by scaling the coordinates.

There are two fixed points of this action: The north pole and the south pole as we act via rotation. At the north pole, the character of the tangent space $T_N\mathbb{P}^1$ is given by $u_1 - u_2$, and as this is only a 1-dimensional thing, we only have 1-character, so this is the same as the Euler class (nothing else to multiply by):

$$\mathrm{char}_N(T_N\mathbb{P}^1) = u_1 - u_2 = e(T_N\mathbb{P}^1)$$

Similarly at the south pole,

$$\mathrm{char}_S(T_S\mathbb{P}^1) = u_2 - u_1 = e(T_S\mathbb{P}^1)$$

Finally, the last thing to do is look at the restriction of the first chern class to these fixed points: But here we just have that $c_1|_N = u_1$ and $c_1|_S = u_2$ (we'll explain below why this is true). As such we have

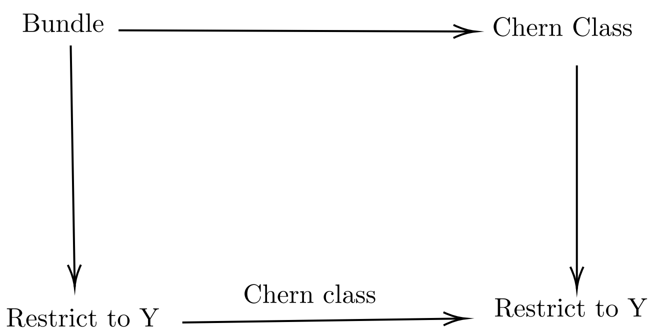
$$\int_{\mathbb{P}^1} c_1 = \frac{c_1|_N}{e(T_N\mathbb{P}^1)} + \frac{c_1|_S}{e(T_S\mathbb{P}^1)} = \frac{u_1}{u_1 - u_2} + \frac{u_2}{u_2 - u_1} = 1$$

Let's explain a bit more about these chern classes.

Assume we have a rank k vector bundle over X, $p : V \to X$, so over each point in X we have a $\mathbb{C}^k$ vector space. Then the chern classes here are elements in the cohomology of X:

$$c_1, c_2, ..., c_k \in H^*(X)$$

With the key property that restriction commutes with taking chern classes. This means that we have the following commutative diagram



Meaning that if we first send our bundle to its chern class, then restrict to Y, we get the same output as first restricting to Y, and then taking a chern class.

For us, this means that

$$c_1\big|_p = c_1(\mathcal{O}(1)_p) = \text{char}(\mathcal{O}(1)_p)$$

Where $\mathcal{O}(1)_p$ is the fiber and a representation of T, of the hyperplane bundle (dual to the tautological bundle). The fiber of the tautological bundle over N/S are coordinate lines, so the characters are just u_1, u_2 (whenever we write c_1 we always mean c_1 of $\mathcal{O}(1)$, and $c_1(\mathcal{O}(1)) = -c_1(\mathcal{O}(-1))$)

Example.

$$H_T^*(\mathbb{P}^{n-1}) = \mathbb{C}[c_1, u_1, ..., u_n]/\text{relations}$$

So the question is what relations? We need to quotient by an ideal, and in particular the ideal of relations should be all possible polynomials that reduce at every fixed point. So this means that we have

$$c_1\big|_p = u_i$$

I.e. the fiber over the ith fixed point is the i^{th} coordinate line scaled by u_i. The relations dictate that we need to quotient by the ideal generated by $(c_1 - u_1)(c_1 - u_2)\cdots(c_1 - u_n)$. Notice that if we sent u_i to 0 we would recover the ordinary cohomology:

$$H^*(\mathbb{P}^{n-1}) = \mathbb{C}[c]/(c^n)$$

7 4/–: Equivariant Integration/27 Lines

8 4/–: Nakajima Quiver Varieties